Non-Homogeneous Solution of Quintic Equation with Five Unknowns $p^4 - q^4 = 82(l^2 - m^2)\sigma^3$

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Abstract

The non-zero unique integral solutions of the Quintic Diophantine equation with five unknowns, expressed as $p^4 - q^4 = 82(l^2 - m^2)\sigma^3$ *can be obtained by applying the linear transformations* $p = u + v$, $q = u - v$; *and* $m = 2u - v$ $(u \neq v \neq 0)$. *Likewise, additional changes can yield several non-zero integer solutions, resulting in multiple integral solutions to the problem. Numerous remarkable relationships between the responses and a few distinct numbers are displayed.*

Keywords: quintic diophantine equation, integral solutions, polygonal numbers, five unknowns

Introduction

An important area of mathematics is the theory of Diophantine equations, a centuries-old subject that usually entails solving polynomial equations in two or more variables or a system of polynomial equations with more unknowns than equations in integers. Diophantine equations deserve special appreciation since they have captivated and motivated mathematicians and enthusiasts alike.

The Diophantine equation's variety [1–4] provides an infinite area of study. A comprehensive list of different quintic equation problems with five unknowns may be found in [4 & 5]. Five-unknown quintic equations are examined in [6–10]. Finding the integer solutions to another intriguing quintic Diophantine equation with five unknowns.

A Quintic equation is a polynomial equation of degree five, meaning the highest power of the variable is five. In this correspondence, the non-homogeneous Quintic condition $p^4 - q^4 = 82(l^2 - m^2)\sigma^3$ with five unknowns addressed by the condition is thought of specifically a couple of fascinating relations among the arrangements are introduced.

Method of Analysis

The non-homogenous quintic equation under consideration as follows:

$$p^4 - q^4 = 82(l^2 - m^2)\sigma^3 \tag{1}$$

Pattern I:

Introducing the linear transformations

$$p=u+v,\ q=u-v;\ l=2u-v \text{ and } m=2u-v\ (u\neq v\neq 0) \quad (2)$$

in (1) leads to

$$u^2+v^2=82\sigma^3 \quad (3)$$

Assume $\sigma=\sigma(a,b)=a^2+b^2$ (4)

where a and b are non-zero distinct integers.

Choice 1: Write 82 as $(9+i)(9-i)$

in (3) becomes, $u+iv=(9+i)(a+ib)^3$

we obtain a system of equations that allows us to separately analyze the real and imaginary components

$$u=9a^3-27ab^2-3a^2b+b^3$$
$$v=a^3-3ab^2+27a^2b-9b^3$$

Employing the values of u and v in (2) we get

$$p=p(a,b)=10a^3-30ab^2+24a^2b-8b^3$$
$$q=q(a,b)=8a^3-24ab^2-30a^2b+10b^3$$
$$l=l(a,b)=19a^3-57ab^2+21a^2b-7b^3$$
$$m=m(a,b)=17a^3-51ab^2-33a^2b+11b^3$$
$$\sigma=\sigma(a,b)=a^2+b^2$$

Properties

- $p(a,1)+q(a,1)-3p_a^3+T_{14,a}+2Gno_a\equiv 0(\bmod 27)$
- $p(a,1)-q(a,1)-SO_a+T_{14,a}+29Gno_a\equiv 0(\bmod 27)$
- $p(a,1)-q(a,1)-l(a,1)+m(a,1)\equiv 0$
- $p(a,1)-q(a,1)-4p_a^5-26CS_a-23Gno_a\equiv 0(\bmod 21)$
- $2l(a,1)-p(a,1)-14p_a^5+4T_{7,a}+45Gno_a\equiv 0(\bmod 51)$
- $l(a,1)-m(a,1)-SO_a-9T_{14,a}-20Gno_a\equiv 0(\bmod 2)$

Choice 2: Write 82 as $(9+i)(9-i)$

(3) implies, $u+iv=(9+i)(a+ib)^3$

Comparing the real and imaginary components, we get the following system of equations:

$$u=a^3-3ab^2-27a^2b+9b^3$$
$$v=9a^3-27ab^2+3a^2b-b^3$$

Substituting u and v in (2) we get

$$p=p(a,b)=10a^3-30ab^2-24a^2b+8b^3$$
$$q=q(a,b)=-8a^3+24ab^2-30a^2b+10b^3$$
$$l=l(a,b)=11a^3-33ab^2-51a^2b+17b^3$$
$$m=m(a,b)=-7a^3+21ab^2-57a^2b+19b^3$$
$$\sigma=\sigma(a,b)=a^2+b^2$$

Properties

- $p(a,1)+q(a,1)-4p_a^5+28T_{6,a}+17Gno_a\equiv 0(\bmod 1)$
- $p(a,1)+q(a,1)+l(a,1)+m(a,1)-3SO_a+23T_{16,a}+2T_{3,a}+76Gno_a\equiv 0(\bmod 22)$
- $2p(1,1)+q(1,1)-2l(1,1)=p(1,1)-3m(1,1)$
- $q(1,1)+m(1,1)-p(1,1)$ is a cubical integer.

The following are perfect squares.

i. $q(1,1)+m(1,1)-p(1,1)-z(1,1)$
ii. $m(1,1)-p(1,1)-q(1,1)=5q(1,1)+p(1,1)$
iii. $2p(1,1)+4q(1,1)-4l(1,1)+m(1,1)$

Pattern II

We can write (3) as $u^2+v^2=82\sigma^3*1$

Let $\sigma=a^2+b^2$

Write 1 as $1=\frac{(3+4i)(3-4i)}{5^2}$ (5)

Using (4) and (5) in (3) and proceeding as above, the corresponding integer solutions to (1) are given below:

$$p=p(a,b)=-50A^3+150AB^2-4800A^2B+1600B^3$$
$$q=q(a,b)=-1600A^3+4800AB^2+150A^2B-50B^3$$
$$l=l(a,b)=-875A^3+2625AB^2-7125A^2B+2375B^3$$
$$m=m(a,b)=-2425A^3+7275AB^2-2175A^2B+725B^3$$
$$\sigma=\sigma(a,b)=25A^2+25B^2$$

Properties

- $32p(a,1)-q(a,1)-l(a,1)-1050P_a^7+7357ST_{6,a}+37925Gno_a\equiv 0(\bmod 10950)$
- $l(a,1)-m(a,1)+\sigma(a,1)-412SO_a-235T_{22,a}-899Gno_a\equiv 0(\bmod 149)$
- $m(a,1)-5T_{12,a}-10Gno_a\equiv 0(\bmod 35)$
- $m(a,1)-q(1,a)-p(1,1)-100P_a^5-955T_{12,a}-1835Gno_a\equiv 0(\bmod 6560)$
- $2[p(a,1)+q(a,1)]=m(a,1)+l(a,1)$
- $\frac{y(1,1)-x(1,1)+p(1,1)^2}{100}$ is a Disarium number is of second order 89.

Pattern III

We can write (3) as $u^2+v^2=82\sigma^3*1$

Write $1=\frac{(5+12i)(5-12i)}{13^2}$ in (3)

$$(u+iv)=\frac{(5+12i)}{13^2}(1+9i)(a+ib)^3$$

Comparing the real and imaginary components on both sides of the equation, leading to further simplification and solution of the problem.

$$u=\frac{1}{13}\left[-103a^3-171a^2b+309ab^2+57b^3\right]$$
$$v=\frac{1}{13}\left[57a^3-309a^2b-171ab^2+103b^3\right]$$

$$\therefore\ p=-7774A^3-81120A^2B+23322AB^2+27040B^3$$
$$q=-27040A^3+23322A^2B+81120AB^2-7774B^3$$
$$l=-25181A^3-110019A^2B+75543AB^2+36673B^3$$
$$m=-44447A^3-5577A^2B+133341AB^2+1859B^3$$
$$\sigma=169p^2+169q^2$$

Properties

- $2[x(1,B)+y(1,B)]-z(1,B)+w(1,B)]\equiv 0$
- $x(1,B)+p(1,B)-1690TO_B-6097T_{38,B}+24258Gno_B\equiv 0(\bmod 21723)$
- $y(1,B)+3887SO_B-21156FT_B-11T_{4,B}-57324Gno_B\equiv 0(\bmod 4976)$
- $w(1,B)-464RD_B-6P_B^5-3403T_{10,B}-T_{4,B}-47329Gno_B\equiv 0(\bmod 3346)$

Conclusion

In this study, we have attempted to solve the non-homogeneous quintic equation with five unknowns, aiming to establish the existence of an infinitely large number of non-zero, distinct integer solutions. Our approach highlights a constructive method that can generate such solutions systematically, demonstrating the richness of the solution space for this class of Diophantine equations. The analysis underscores the complexity and potential of higher-degree equations in multiple variables, offering a new perspective on their solvability. Building on these findings, future investigations can focus on exploring broader families of higher-degree Diophantine equations involving more variables. Techniques such as parametric representations, transformation of variables, and the use of computational number theory tools can be instrumental in identifying new infinite solution sets. This direction not only contributes to the understanding of quintic forms but also opens new paths for solving complex equations in number theory and algebraic structures.

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