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Enticing Sequences of Polynomial Diophantine Triple

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Abstract

This paper examines the establishment of an infinite sequences of polynomial Diophantine triples and substantiates that these sequences comprising of triplets is regular and cannot be extended into a quadruple. The findings are derived from the Diophantine pair formed by Vieta polynomials, specifically the Vieta-Pell and Vieta-Pell-Lucas polynomial. The numerical values are meticulously calculated using MATLAB.

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Introduction

Diophantine triples have a rich history, beginning with Diophantus of Alexandria's exploration of integer solutions in polynomial equations. Over the centuries, mathematicians such as Pierre de Fermat and Leonhard Euler expanded on his foundational work, discovering numerous Diophantine triples and furthering the field of number theory. Among these, Polynomial Diophantine tuples hold a prominent position in it.

A polynomial Diophantine *k*-tuple $\{a_1(y), a_2(y), a_3(y), \dots\dots\dots a_k(y)\}$ is a set of non-zero polynomial with integer coefficient that possess the unique property where the product of any two polynomials a_i and a_j plus 1 yields a perfect square, for all $1 \leq i < j \leq k$. Wherein, a $D(m) - k -$ polynomial tuples, which are sets of polynomials with integer coefficient $\{a_1(y), a_2(y), a_3(y), \dots\dots\dots a_k(y)\}$ have the property that the product of any two polynomials of the set increased by gives square of another polynomial. There are numerous polynomial Diophantine tuples holding different property which can be explored in [1-7].

A polynomial Diophantine triple $(a_1(y), a_2(y), a_3(y))$ is regular if $(a_3(y) - a_2(y) - a_1(y))^2 = 4(a_1(y)a_2(y) + m)$.

In this article, sequences of regular polynomial Diophantine triple protracted from Diophantine pair employing Vieta-Pell, Vieta-Pell Lucas polynomials are established. Extensibility of these sequences are examined and numerical illustrations are obtained with the aid of MATLAB.

Sequences of Vieta Polynomials

In this section, we establish Polynomial Diophantine triple sequences from Vieta polynomials by algebraic manipulations.

Vieta-Pell polynomial

The Vieta-Pell polynomial is defined as

$$\Delta_n(y) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 2y\Delta_{n-1}(y) - \Delta_{n-2}(y), & n \geq 2 \end{cases}$$

Consider the following two Vieta-Pell polynomials $p_1(y) = \Delta_2(y) = 2y$ and $p_2(y) = \Delta_4(y) = 8y^3 - 4y$. It is noted that, $(p_1(y), p_2(y))$ builds a Diophantine pair as the product of these polynomials increased by one result in $(4y^2-1)^2$.

With the purpose of extending this pair to triple choose $p_3(y)$ as third tuple, thus

$$p_1(y) * p_3(y) + 1 = \alpha_1^2$$

$$p_2(y) * p_3(y) + 1 = \alpha_2^2$$

Voiding $p_3(y)$

$$(p_2(y) - p_1(y)) = \alpha_1^2 p_2(y) - \alpha_2^2 p_1(y)$$

Initiating the linear conversions as $\alpha_1 = \theta_1 + p_1(y)\theta_2$, $\alpha_2 = \theta_1 + p_2(y)\theta_2$. The above expression turns into a notable Pell equation

$$\theta_1^2 = D\theta_2^2 + 1$$

where $D=16y^4-8y^2$, which does not form a perfect square polynomial and the fundamental solution of the Pell equation is $(\theta_1, \theta_2) = (4y^2 - 1, 1)$.

By applying the fundamental solution to one of the linear conversions $(say\ \alpha_1)$ and substituting α_1, $p_1(y)$, third polynomial emerges

$$p_3(y) = 8y^3 + 8y^2 - 2y - 2$$

As a result, $(p_1(y), p_2(y), p_3(y))$ is the polynomial Diophantine triple extended from pair.

Correspondingly, consider the Diophantine pair $(p_1(y), p_2(y), p_3(y))$. Adhering to the previously mentioned steps, a Diophantine triple $(p_2(y), p_3(y), p_4(y))$ is generated with

$$p_2(y) = 8y^3 - 4y$$

$$p_3(y) = 8y^3 + 8y^2 - 2y - 2$$

$$p_4(y) = 32y^3 + 16y^2 - 14y - 4$$

$$(p_2(y), p_3(y), p_4(y))$$

Hence, $(p_2(y), p_3(y), p_4(y))$ is another polynomial Diophantine triple.

Utilizing this method an infinite sequence of polynomial Diophantine triples

$$\{(2y, 8y^3 - 4y, 8y^3 + 8y^2 - 2y - 2), (8y^3 - 4y, 8y^3 + 8y^2 - 2y - 2, 32y^3 + 16y^2 - 14y - 4), \dots\dots, \dots\dots, \dots\}$$

is established.

Vieta-Pell Lucas polynomial

The Vieta-Pell Lucas polynomial is defined as

$$\nabla_n(y) = \begin{cases} 2, & n = 0 \\ 2y, & n = 1 \\ 2y\nabla_{n-1}(y) - \nabla_{n-2}(y), & n \geq 2 \end{cases}$$

Consider two Vieta-Pell Lucas polynomials $l_1(y) = \nabla_1(y) = 2y$ and $l_2(y) = \nabla_3(y) = 8y^3 - 6y$. $(l_1(y), l_2(y))$ forms a Diophantine pair, as their product when added to a polynomial $D(4 - 4y^2)-$, results in perfect square of another polynomial.

Now, for extending it to triple $(l_1(y), l_2(y), l_3(y))$, perform the algebraic manipulations outlined above. This process yields the third polynomial tuple

$$l_3(y) = 8y^3 + 8y^2 - 4y - 4$$

Thus, $(l_1(y), l_2(y), l_3(y))$ is a $D(4 - 4y^2)-$ polynomial Diophantine triple.

Similarly, another triple $(l_2(y), l_3(y), l_4(y))$ is derived from the Diophantine pair $(l_2(y), l_3(y))$ with

$$l_2(y) = 8y^3 - 6y$$

$$l_3(y) = 8y^3 + 8y^2 - 4y - 4$$

$$l_4(y) = 32y^3 + 16y^2 - 22y - 8$$

Therefore, $(l_2(y), l_3(y), l_4(y))$ is another $D(4 - 4y^2)-$ polynomial triple. As a result, this method establishes an infinite number of sequences of such polynomial triples

$$\{(2y, 8y^3 - 6y, 8y^3 + 8y^2 - 4y - 4), (8y^3 - 6y, 8y^3 + 8y^2 - 4y - 4, 32y^3 + 16y^2 - 22y - 8), \dots\dots, \dots\dots, \dots\}$$

Regularity

The triple $(p_1(y), p_2(y), p_3) = (2y, 8y^3 - 4y, 8y^3 + 8y^2 - 2y - 2)$ is regular as

$$(8y^2 - 2)^2 = 4(16y^4 - 8y^2 + 1)$$

Similarly, the triples $(p_2(y), p_3(y), p_4(y))$, $(l_1(y), l_2(y), l_3(y))$, $(l_2(y), l_3(y), l_4(y))$ also exhibit regularity, adhering to the condition.

Extensibility

To extend the triples to quadruples choose the fourth polynomial tuple as $p(y)$ in $(p_1(y), p_2(y), p_3(y))$ such that,

$$\beta_1(y) * \beta(y) + 1 = \alpha_3^2$$

$$\beta_2(y) * \beta(y) + 1 = \alpha_4^2$$

$$\beta_3(y) * \beta(y) + 1 = \alpha_5^2$$

Adhering to the same criteria, the value of fourth polynomial tuple is

$$\beta(y) = 8y^3 + 16y^2 + 4y - 4$$

However, substituting $\beta(y)$ into any of the above equations does not satisfy the requirement, as the result is not a polynomial square. Thus, it does not form a polynomial quadruple.

Likewise, it is observed that $(\beta_2(y), \beta_3(y), \beta_4(y))$, $(l_1(y), l_2(y), l_3(y))$ and $(l_2(y), l_3(y), l_4(y))$ are inextensible.

Numerical illustrations

Figure 1 Depicts Numerical values for Regular Polynomial Diophantine triple sequences identified in MATLAB.

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MATLAB Command Window

Vieta Polynomial Diophantine triple Sequences
Vieta Pell polynomial Diophantine triple sequence
2       4       12      4       12      30
4       56      90      56      90      288
6       204     280     204     280     962
8       496     630     496     630     2244
Vieta Pell Lucas Polynomial Diophantine triple sequence
for y=1 triple does not exist
4       52      84      52      84      268
6       198     272     198     272     934
8       488     620     488     620     2208
10      970     1176    970     1176    4282
>>
```

Figure 1

Conclusion

An enticing sequences of polynomial Diophantine triples $\{(\beta_1(y), \beta_2(y), \beta_3(y)), (\beta_2(y), \beta_3(y), \beta_4(y)), \ldots\}$ $\{(l_1(y), l_2(y), l_3(y)), (l_2(y), l_3(y), l_4(y)), \ldots\}$, utilizing Vieta polynomials namely, Vieta-Pell and Vieta-Pell Lucas polynomials are established. This work opens up new avenues for further exploration with other polynomials using MATLAB.

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